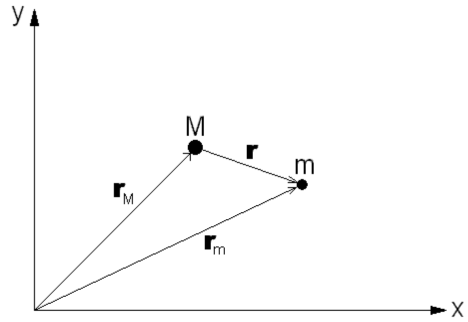
Determination of the velocity of a celestial body in the space



$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\dot{\mathbf{r}} = \frac{d}{dt}\mathbf{r} \qquad \ddot{\mathbf{r}} = \frac{d}{dt}\dot{\mathbf{r}}$$

$$\mathbf{F}_m = m \cdot \ddot{\mathbf{r}}_m = +G \cdot \frac{M \cdot m}{r^2} \cdot \frac{\mathbf{r}_M - \mathbf{r}_m}{r} \qquad 1a)$$

$$\mathbf{F}_M = M \cdot \ddot{\mathbf{r}}_M = -G \cdot \frac{M \cdot m}{r^2} \cdot \frac{\mathbf{r}_M - \mathbf{r}_m}{r} \qquad 1b)$$

$\mathbf{F}$ is the force on the body and G the gravitational constant

$$\mathbf{r} = \mathbf{r}_m - \mathbf{r}_M$$

after substraction of $\ddot{\mathbf{r}}_m$ and $\ddot{\mathbf{r}}_M$ obtained from 1a) and 1b)

$$\ddot{\mathbf{r}} = +\frac{G}{r^3} M(\mathbf{r}_M - \mathbf{r}_m) + \frac{G}{r^3} m(\mathbf{r}_M - \mathbf{r}_m) = -G \cdot (M + m)\frac{\mathbf{r}}{r^3}$$

if we multiply vectorial with $\mathbf{r}$ it succeeds:

$$\mathbf{r} \times \ddot{\mathbf{r}} = -G \cdot (M + m) \cdot \frac{\mathbf{r} \times \mathbf{r}}{r^3} = 0$$

on the other hand it is valid:

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r} \times \ddot{\mathbf{r}} + \dot{\mathbf{r}} \times \dot{\mathbf{r}} = 0$$

Integration provides:

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C}$$

$\mathbf{C}$ is the vector for the impuls of rotation
after vectorial multiplication with with $\ddot{\mathbf{r}}$

$$\mathbf{C} \times \ddot{\mathbf{r}} = -\frac{G \cdot (M + m)}{r^3} \cdot (\mathbf{C} \times \mathbf{r}) = -\frac{G \cdot (M + m)}{r^3} \cdot ((\mathbf{r} \times \dot{\mathbf{r}}) \times \mathbf{r})$$

$$= -\frac{G \cdot (M + m)}{r^3} \cdot (\dot{\mathbf{r}} \cdot (\mathbf{r} \cdot \mathbf{r}) - \mathbf{r} \cdot (\mathbf{r} \cdot \dot{\mathbf{r}}))$$

Because of :

$$\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \mathbf{r}\frac{d}{dt}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{d}{dt}(\mathbf{r}) = \mathbf{r}\left(-\frac{1}{r^2}\dot{r}\right) + \frac{1}{r}\dot{\mathbf{r}} =$$

$$= \dot{\mathbf{r}}\frac{1}{r} - \mathbf{r}\frac{\dot{r}}{r^2} = \frac{1}{r^3}(\dot{\mathbf{r}}(\mathbf{r} \cdot \mathbf{r}) - \mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}))$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\dot{r} = \frac{1}{2}\frac{2x\dot{x} + 2y\dot{y} + 2z\dot{z}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}\cdot\dot{\mathbf{r}}}{r}$$

$$r\cdot\dot{r} = \mathbf{r}\cdot\dot{\mathbf{r}}$$

With it it is valid:

$$\mathbf{C}\times\ddot{\mathbf{r}} = -G\cdot(M+m)\cdot\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

Integration with (-$\mathbf{A}$) as integration constant provides

$$\mathbf{C}\times\dot{\mathbf{r}} = -G\cdot(M+m)\cdot\frac{\mathbf{r}}{r} - \mathbf{A} \qquad 2)$$

Because of

$$\mathbf{C}\cdot(\mathbf{C}\times r) = r\cdot(\mathbf{C}\times\mathbf{C}) = 0 =$$

$$= -G\cdot(M+m)\frac{1}{r}\cdot(\mathbf{C}\cdot r) - (\mathbf{C}\cdot\mathbf{A}) =$$

$$= -(\mathbf{C}\cdot\mathbf{A}) \qquad \mathbf{C} \text{ perpendicular to } \mathbf{A}$$

If we multiply 2) with $\mathbf{r}$ than we obtain:

$$((\mathbf{C}\times\dot{\mathbf{r}})\cdot\mathbf{r}) = ((\dot{\mathbf{r}}\times\mathbf{r})\cdot\mathbf{C}) = -\mathbf{C}\cdot\mathbf{C} = -C^2$$

$$= -G\cdot(M+m)\frac{1}{r}(\mathbf{r}\cdot\mathbf{r}) - \mathbf{A}\cdot\mathbf{r}$$

or with ν as the angle between $\mathbf{A}$ and $\mathbf{r}$:

$$C^2 = -G\cdot(M+m)\cdot r + A\cdot r\cdot\cos\nu$$

Do we define the magnitudes p (orbit parameter) and e (excentricity) as

$$p = \frac{C^2}{G\cdot(M+m)} \qquad e = \frac{A}{G\cdot(M+m)}$$

then it is valid:

$$p = r \cdot (1 + e \cdot \cos\nu) \qquad r = \frac{p}{1 + e \cdot \cos\nu}$$

Do we choose the motion in the plane we obtain:

$$C = x \cdot \dot{y} - y \cdot \dot{x} \qquad 3)$$

$$x = r \cdot \cos\nu \qquad \dot{x} = -r \cdot \sin\nu \cdot \dot{\nu} + \dot{r} \cdot \cos\nu$$

$$y = r \cdot \sin\nu \qquad \dot{y} = +r \cdot \cos\nu \cdot \dot{\nu} + \dot{r} \cdot \sin\nu$$

$C = r^2\dot{\nu}$ we obtain after inserting 3)

$$C = \sqrt{G \cdot (M + m) \cdot p}$$

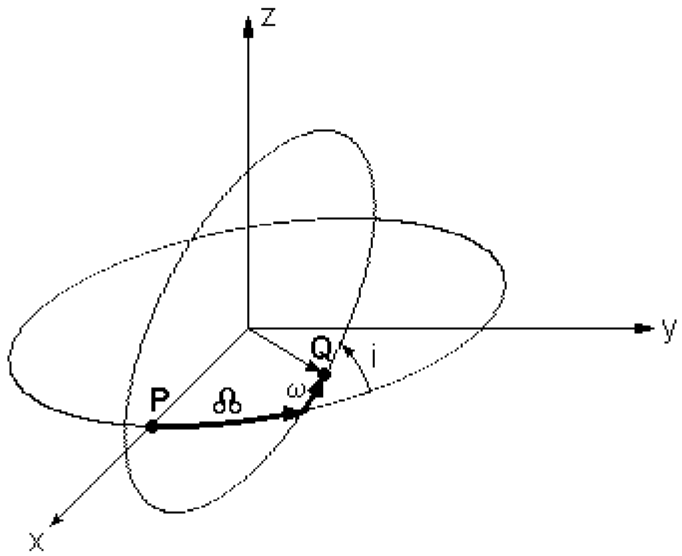
$$\dot{\nu} = \sqrt{\frac{G \cdot (M + m)}{p^3}} \cdot (1 - e \cdot \cos\nu)^2$$

$$\dot{r} = \frac{p}{(1 + e \cdot \cos\nu)^2} \cdot e \cdot \sin\nu \cdot \dot{\nu} = \sqrt{\frac{G \cdot (M + m)}{p}} \cdot e \cdot \sin\nu$$

$$\dot{x} = -\sqrt{\frac{G \cdot (M + m)}{p}} \cdot \sin\nu$$

$$\dot{y} = +\sqrt{\frac{G \cdot (M + m)}{p}} \cdot (e + \cos\nu)$$

now we transform the plane position into the space:



$$\mathbf{P} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \mathbf{Q} = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}$$

$$\mathbf{Q} = Z_\Omega \cdot X_i \cdot Z_\omega \cdot \mathbf{P}$$

$$\mathbf{v} = Z_\Omega \cdot X_i \cdot Z_\omega \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix}$$

$$\mathbf{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \sqrt{\frac{G \cdot (M + m)}{p}} \cdot Z_\Omega \cdot X_i \cdot Z_\omega \cdot \begin{pmatrix} -\sin\nu \\ e + \cos\nu \\ 0 \end{pmatrix}$$

$$Z_{\angle} = \begin{pmatrix} \cos\angle & -\sin\angle & 0 \\ \sin\angle & \cos\angle & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Matrix of turning around the z - axis

$$X_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}$$ Matrix of turning around the x - axis

v is now the vector of velocity in space

now we only need the calculation of the true anomaly ν:

in case of an elliptic orbit:

$$tg\left(\frac{\nu}{2}\right) = \sqrt{\frac{1+e}{1-e}} \cdot tg\left(\frac{E}{2}\right)$$ ν = true anomaly

$$E = M + e \cdot \sin E$$ E = excentric anomaly

$$M = \frac{2\pi}{T} \cdot (t - t_0)$$ M = medium anomaly

t_0 = time of perihelion passage when $\nu = 0$

$$T = 2\pi\sqrt{\frac{a^3}{G \cdot (M+m)}}$$ T = encircling time

a = major semiaxis

b = minor semiaxis